3D Transformations

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3D Transformations

- Methods for geometric transformation are extended from 2D to 3D by including considerations for the z-coordinate.
- In case of translation, the three dimensional translation vector specify that how much an object has been translated in three coordinate direction.

3D Transformation

- Transformation Matrix

$$\begin{bmatrix} A & D & G & J \\ B & E & H & K \\ C & F & I & L \\ 0 & 0 & 0 & S \end{bmatrix} \Rightarrow \left[\begin{array}{c|c} 3\times 3 & 3\times 1 \\ \hline 1\times 3 & 1\times 1 \end{array}\right]$$

3×3 : Scaling, Reflection, Shearing, Rotation

3×1 : Translation

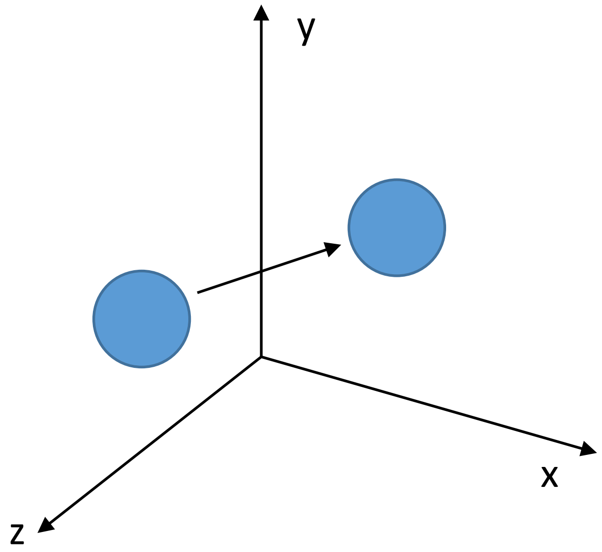
1×1 : Uniform global Scaling

1×3 : Homogeneous representation

3D Translation

- **Translation of a Point : P'=T.P**

$$x' = x + t_x, \quad y' = y + t_y, \quad z' = z + t_z$$

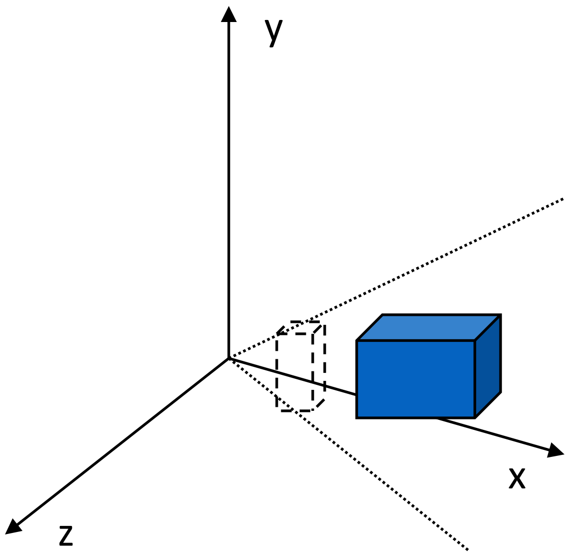


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

3D Scaling

- **3D Scaling : P’=S.P**

$$x' = x \cdot s_x \,, \quad y' = y \cdot s_y \,, \quad z' = z \cdot s_z$$

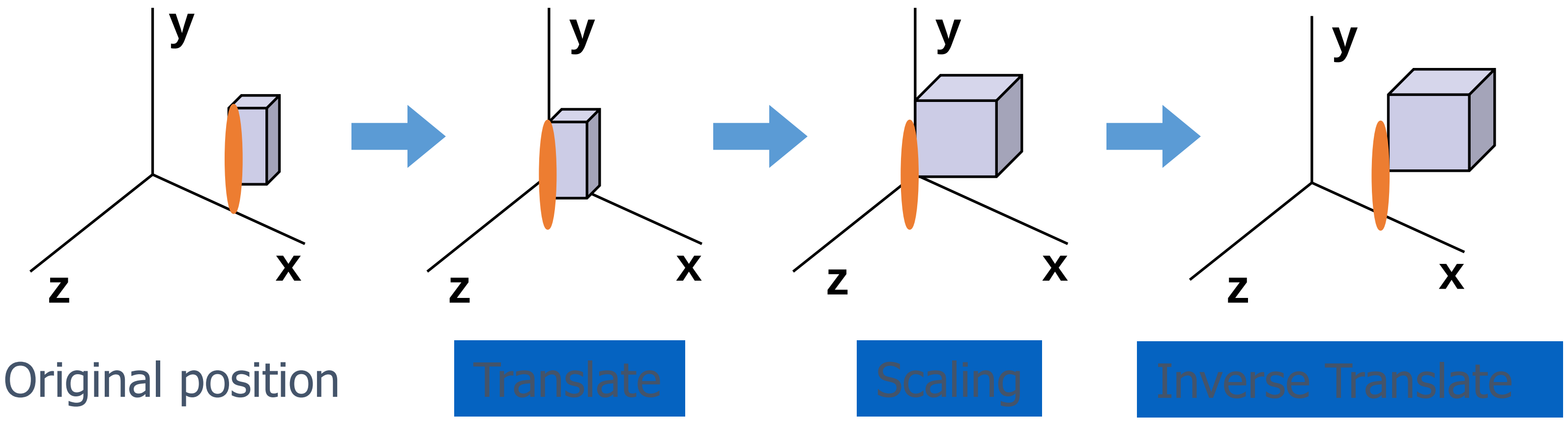


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Relative Scaling

- **Scaling with a Selected Fixed Position(x_f, y_f, z_f)**

1. **Translate the fixed point to the origin**
2. **Scale the object relative to the coordinate origin.**
3. **Translate the fixed point back to its original position.**



$$T(x_f, y_f, z_f)\cdot S(s_x, s_y, s_z)\cdot T(-x_f, -y_f, -z_f) = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & x_f \\ 0 & 1 & 0 & y_f \\ 0 & 0 & 1 & z_f \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -x_f \\ 0 & 1 & 0 & -y_f \\ 0 & 0 & 1 & -z_f \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} S_x & 0 & 0 & (1-S_x)x_f \\ 0 & S_y & 0 & (1-S_y)y_f \\ 0 & 0 & S_z & (1-S_z)z_f \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3D Rotation

- To generate a rotation transformation for an object

→An axis about which the object is to be rotated and amount of angular rotation is required.

→Positive rotation directions about the coordinate axis are counter clockwise when looking towards the origin from a position on each axis.

- Coordinate-Axes Rotations
  - X-axis rotation
  - Y-axis rotation
  - Z-axis rotation
- General 3D Rotations
  - Rotation about an axis that is parallel to one of the coordinate axes
  - Rotation about an arbitrary axis

Coordinate-Axes Rotations

Z-Axis Rotation

$x' = x\cos\theta - y\sin\theta$

$y' = x\sin\theta + y\cos\theta$

$z' = z$

X-Axis Rotation

$x' = x$

$y' = y\cos\theta - z\sin\theta$

$z' = y\sin\theta + z\cos\theta$

Y-Axis Rotation

$x' = z\sin\theta + x\cos\theta$

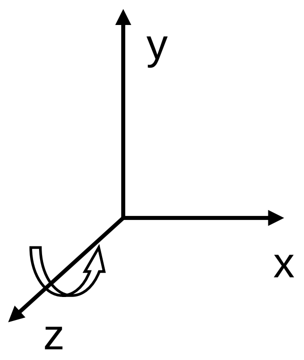
$y' = y$

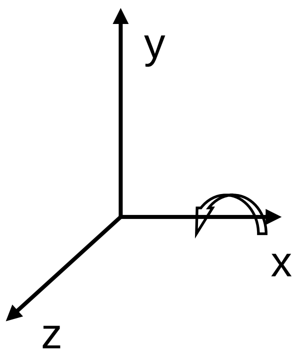
$z' = z\cos\theta - x\sin\theta$

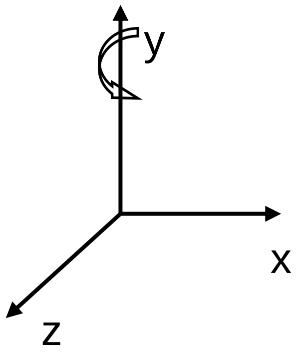
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

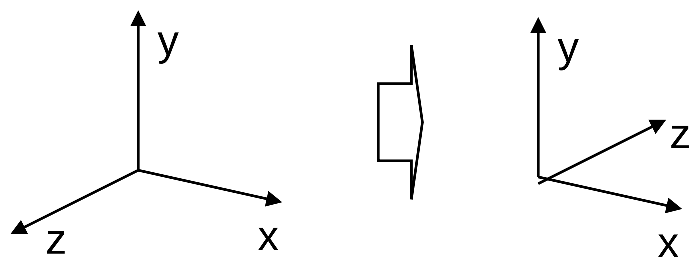






Other Transformations

- **Reflection Relative to the xy Plane**



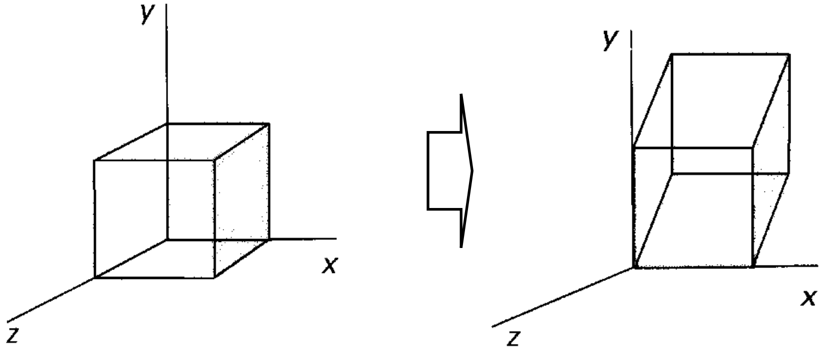
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- **Reflection can be performed relative to a selected reflection axis and it is equivalent to 180^0 rotation in four dimensional space (xy, yz, xz)**
- **The matrix representation reflection of a point relative to xy plane is shown above.**
- **In yz plane** $R_{fx} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **In xz plane** $R_{fy} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Other Transformations

- **Z-axis Shear: After x, y coordinate without changing z coordinate.**

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



- **X-axis Shear:**

$$SHx = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Y-axis Shear:

$$SHy = \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Q: Translate the point P(5,6,7) by 3 co-ordinates in x-direction, 3 co-ordinates in y-direction and 2- co-ordinates in z-direction and find the new co-ordinates of P i.e P'.

Solution: Given P(5,6,7)

$t_x = 3$, $t_y = 3$, $t_z = 2$

$\therefore$ Translation matrix is

$$T = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$P' = T.P$

$$\therefore \quad P' = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 5+3 \\ 6+3 \\ 7+2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \\ 9 \\ 1 \end{bmatrix}$$

$\therefore \; P'(x', y, z') = P'(8, 9, 9)$ Ans.

Q: Given a 3D object with co-ordinate points A(0, 3, 3), B(3, 3, 6), C(3, 0, 1), D(0, 0, 0). Apply the scaling parameter 2 towards X-axis, 3 towards Y-axis and 3 towards Z-axis and obtain the new co-ordinates of the object.

Solution Given A(0, 3, 3), B(3, 3, 6), C(3, 0, 1), D(0, 0, 0)

Given $S_x = 2$, $S_y = 3$, $S_z = 3$

$\therefore$ Scaling matrix is

$$S = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\therefore$

$$[A' \ B' \ C' \ D'] = S \times [A \ B \ C \ D]$$

$$= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 3 & 3 & 0 \\ 3 & 3 & 0 & 0 \\ 3 & 6 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 6 & 6 & 0 \\ 9 & 9 & 0 & 0 \\ 9 & 18 & 3 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$\therefore$ $A'(0,9,9)$, $B'(6,9,18)$, $C'(6,0,3)$, $D'(0,0,0)$

Q: Given a homogeneous point (1, 2, 3). Apply 90° rotation towards X, Y and Z axis and find out the new co-ordinate points.

Solution: Given $P(1,2,3)$, $\theta = 90^\circ$

For X-axis Rotation

3D Rotation Matrix for x-axis rotation

$$R_{\theta x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow R_{x90^\circ} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\therefore P' = R_{x\theta} \cdot P$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 2 \\ 1 \end{bmatrix}$$

$\therefore$ New co-ordinates after rotation along x-axis $= (1, -3, 2)$

For Y-axis Rotation

3D Rotation Matrix for y-axis rotation

$$R_{y\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow R_{y90^\circ} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore P' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -1 \\ 1 \end{bmatrix}$$

$\therefore$ New co-ordinates after rotation along y-axis $= (3, 2, -1)$

For Z-axis Rotation

3D Rotation matrix for z-axis rotation

$$R_{z\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow R_{z90^\circ} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore P' = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}$$

$\therefore$ New co-ordinates after rotation along z-axis $= (-2, 1, 3)$

Q:- Given a block ABCDEFGH with the following position vector

$$[X] = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 2 & 0 & -1 & 1 \\ 2 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & -2 & 1 \\ 2 & 1 & -2 & 1 \\ 1 & 1 & -2 & 1 \end{bmatrix}$$

Find the reflection of the given block in xy Plane.

Solution

Reflection Matrix in xy-plane

$$R_{fz} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\therefore \quad [X'] = [X][R_{fz}]$

$$= \begin{bmatrix} 1 & 0 & -1 & 1 \\ 2 & 0 & -1 & 1 \\ 2 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & -2 & 1 \\ 2 & 1 & -2 & 1 \\ 1 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[X'] = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 \\ 2 & 0 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 \end{bmatrix}$$

$\therefore$ $[X']$ represent the new Reflection co-ordinates of the given block ABCDEFGH ~~after~~ in xy plane.

A'(1,0,1), B'(2,0,1), C'(2,1,1), D'(1,1,1), E'(1,0,2), F'(2,0,2), G'(2,1,2), H'(1,1,2)

Q: Consider a unit cube. Apply the given shearing transformation Sh on the unit cube and calculate the new co-ordinates of the unit cube after shearing.

$$Sh = \begin{bmatrix} 1 & -0.85 & 0.25 & 0 \\ -0.75 & 1 & 0.7 & 0 \\ 0.5 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution

Co-ordinate matrix for the unit cube

$$[X] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\therefore [X'] = [X][Sh]$$

$$= \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -0.85 & 0.25 & 0 \\ -0.75 & 1 & 0.7 & 0 \\ 0.5 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[X'] = \begin{bmatrix} 0.5 & 1 & 1 & 1 \\ 1.5 & 0.15 & 1.25 & 1 \\ 0.75 & 1.15 & 1.95 & 1 \\ -0.25 & 2 & 1.7 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & -0.85 & 0.25 & 1 \\ 0.25 & 0.15 & 0.95 & 1 \\ -0.75 & 1 & 0.7 & 1 \end{bmatrix}$$

$\therefore$ $[X']$ is the new co-ordinates of the unit cube after applying shearing.